Nonlinear Analysis of Financial Time Series

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Abstract

One of the axioms of the modern science states that, if one can identify an exact mathematical description of a physical system, then a very detailed understanding of the system's properties is possible. A very accurate prediction of its future behavior would be also possible.

These assertions proved to be true only for particular cases and false for nonlinear systems. The vast majority of natural phenomenon has a nonlinear behavior, completely different from the idealized linear dynamics. It's very clear that, an adaptation of the methods used for linear systems analysis is not possible and hence the need for a new mathematical apparatus.

The paper aims at explaining these concepts and to analyze the behavior of two time series, one corresponding to the currency exchange rate (Leu - Euro) and the other one to the Cambridge Bitcoin Electricity Consumption Index (CBECI).

Key words: nonlinear analysis, chaos, neural networks, ARIMA
J.E.L. classification: C02, C40

1. Introduction

Until recently, linear models, which are easier to understand, were used in the majority of modelling problems. Linear theory considered that a complex and irregular behavior is due to intricate effect of external component that affects system dynamic, making very difficult or impossible to predict its evolution.

Towards the end of the past century, nonlinear models developed due to their superior modelling performances.

Nonlinear behavior allows a better understanding of the complex natural phenomena. Nonlinear dynamics introduced new set of tools and concepts used to analyze and to investigate the dynamic generated by nonlinear processes. One can state that, at the moment, a conceptual unification of the notions (period doubling, bifurcations, attractors, initial condition sensitivity, Lyapunov exponent) has been accomplished. The techniques and methods researching the concepts introduced by the nonlinear dynamics are collected under the name of nonlinear signal processing or nonlinear analysis.

Each state of a deterministic dynamical system is the outcome of a set of rules governing its dynamics (excepting pure random systems), even if these rules cannot be easily identified or identified at all due to the extreme complexity of the system.

The behavior of a dynamical nonlinear system is illustrated in the phase (or state) space - a conceptual space whose dimensions are corresponding to the system's variables. The evolution in time of the system described by differential equations is reflected in the moves (trajectories) of a point in the state space. The phase portrait is an image of the system's state change at different moments in time. Phase portraits may reveal the existence of regions or points towards which all the neighboring trajectories are converging. A strange attractor will occupy a region of the state space that captures all the trajectories which, apparently following a randomly pattern, will cover its surface without repeating themself.

Nonlinear analysis is based on Taken's state space reconstruction theorem and the concept of topological equivalence. When working with a reconstructed low dimension phase space, the main features of the original dynamics are replicated, without necessarily knowing the equations of the dynamical system that generates the time series.

The concept of topological equivalence facilitates the study of the low dimensional geometrical objects. The study will reveal information regarding the original dynamics. Topological equivalence and low dimension attractor are leading to another important concept – capacity dimension which discriminates between deterministic chaos and random behavior. The methodology of deterministic chaos identification will be addressed later in the paper.

The paper aim is to reveal the chaotic dynamics (if any) in the leu – euro exchange rate and CBECI (Cambridge Bitcoin electricity Consumption Index) time series. We will transform the time series in order to meet an important request, the stationarity, and then we will perform a test which will confirm that the time series meet this criterion in order to proceed with the nonlinear analysis methodology. Only relevant result will be mentioned in the paper.

2. Theoretical background

In the last 20 years, different signal analysis and processing techniques were set up. These techniques are mainly based on Taken's state space reconstruction theorem. Taken's theorem allows the reconstruction of the original state space without knowing the exact model of the system.

In order to correctly reconstruct the original state space, the embedding dimension is of capital importance. Let us consider a discrete dynamical system defined by:

$$x_{t+1} = f(x_t) \tag{1}$$

Where $x_t \in R^n$ is a state vector and $f : R^n \to R^n$ is a smooth and continuously differentiable function in the n dimensional space. It is very difficult to observe all x_t. To reconstruct an equivalent state space, a single variable is considered with a certain delay. Taken's theorem states that knowing the value of a single variable describing the system's state is enough to reconstruct the original state space.

State space reconstruction principle is that any property of a system based on the distance between two points will be preserved when the correct embedding dimension will be attained.

If the evolution of the dynamical system is described by the time series $\{x_1, x_2, \ldots, x_N\}$, then the reconstruction of the dynamics starting from the x_i variable observed values is:

$$\{y_1, y_2, \ldots, y_{N-(m-1)\tau}\} \tag{2}$$

The vectors $y_1 \ldots y_{N-(m-1)\tau}$ are called the delay vectors, The trajectory resulting after the reconstruction is:

$$y_k = \{x_k, x_{k+\tau}, x_{k+2\tau}, \ldots, x_{k+(m-1)\tau}\} \tag{3}$$

Where τ is the delay and m is the embedding dimension. An embedding dimension equal to 1 is equivalent with the original dynamic an gives the original times series. A right choice for the delay and the embedding dimension guarantee that the attractor is completely unfolded and it is topologically identical with the original one in the phase space.

It has been proved according to Sauer's theorem that the embedding dimension m has to verify the inequality:

$$m \geq 2d_a + 1 \tag{4}$$

Where d_a is the attractor's dimension (the dimension of the state space) and m has an integer value.

If the diameter of the reconstructed attractor is d, then the neighbors of the reference points used to measure the divergence of the neighboring trajectories are included in a r radius sphere, r<d. The ratio of the two quantities is ρ, $\rho \leq 0.1$. The length of the time series is given by:

$$log\, N > d_a\, log\left(\frac{1}{\rho}\right) \tag{5}$$

where N is the number of realizations of the time series and d_a is the attractor dimension. If ρ=0.1 then

$$N > 10^D \tag{6}$$

2.1 The embedding dimension estimation

The accuracy of the embedding dimension estimation process is important because the embedding dimension indicates how many differential equations will be needed to model the behavior of the system under study, or, if an ANN is used for modelling purposes, indicates the number of the inputs, e. q. the number of the neurons in the input layer.

After the embedding dimension is correctly determined, the attractor will be completely unfolded. A higher dimension will alter its geometry.

An incorrect embedding dimension will influence the modelling in the following directions:

- The number of points of the attractor could be insufficient for computing some invariants, for instance the Lyapunov exponent;
- An embedding dimension higher than the correct one will not contain useful information about the attractor which is already included in a smaller embedding dimension;
- The computing complexity increases with the increase of the embedding dimension

The methods used for finding the minimum embedding dimension are: false nearest neighbors method, attractor invariants saturation method and the singular value decomposition method.

The method of false nearest neighbors is based on the idea that if the embedding dimension is incorrectly determined, the points that are closer in the reconstructed space were placed farther in the original state space. Hence, in the reconstructed state space based on an incorrect embedding dimension the neighbors are false nearest neighbors.

An exact identification of the embedding dimension is carried out in practice by representing the percentage of false nearest neighbors vs the embedding dimension. When the percentage drops to 0, the attractor is considered to be completely unfolded.

The method of attractor's invariants saturation relies on the property that, if is completely unfolded, the fractal dimension (or the correlation dimension) of the attractor is independent of embedding dimension. Saturation of the correlation dimension is one of the properties of chaotic systems. The fractal dimension of a strange attractor from (usually) an one dimensional time series is determined using Grassberger and Procaccia algorithm, (Grassberger, 1983).

Once determined, we will know that the minimum embedding dimension is the integer value greater than the correlation dimension.

$$m > [d_a] \tag{7}$$

A generalization of Takens theorem was obtained by Sauer. Sauer's theorem which states that, when an attractor with d_a fractal dimension is projected in a dimension $m>2d_a+1$, all self-crossing of attractor's trajectories are removed.

2.2. The delay parameter

Despite the fact that, in theory, the value of the delay parameter is not very important, the graphical representation of the time series depends totally on it. The delay parameter influences essentially the attractor reconstruction: if the delay is too low, then the attractor is compressed along the main diagonal of the phase space. It the delay is too high, the original dynamic will not be reconstructed accurately, resulting a very intricate dynamic, even if the original dynamic is very simple.

In order to find the delay, two methods are used: the method of autocorrelation function and the method of mutual information.

The mutual information function can be considered a generalization of the autocorrelation function, measuring the linear dependence mutual information for all the points, when the measurements are delayed in time. The average mutual information is the average quantity of information of $y(t+\tau)$ knowing $y(t)$.

Mutual information is determined using the following equation:

$$I(T) = \sum_{n=1}^{N} P(y_n, y_{n+T}) \, log_2 \frac{P(y_n, y_{n+T})}{P(y_n)P(y_{n+T})} \tag{8}$$

where $P(y_n,y_{n+T})$ is the probability of observing y_n, and $P(y_n)$ represents the probability of observing y_{n+T}. I(T) represents the quantity of information available about y_n when observing y_{n+T}..

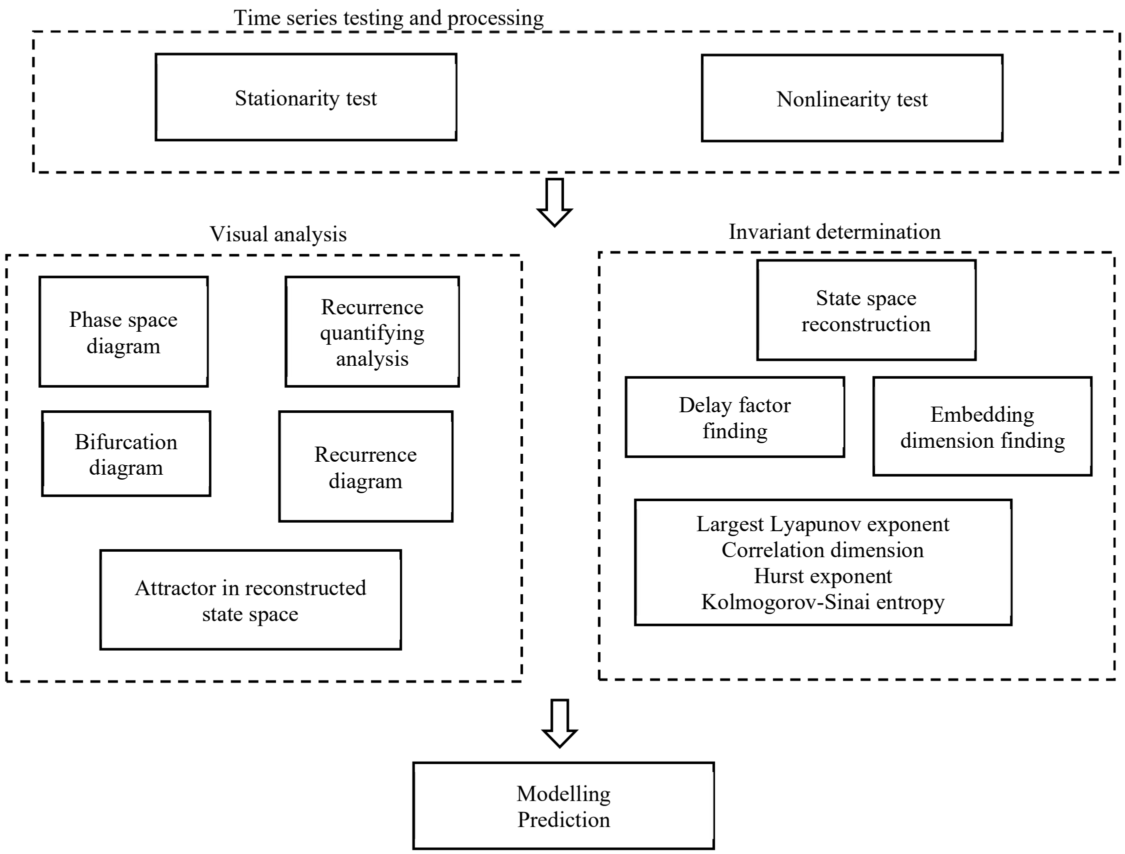
The most used method is the mutual information method because nonlinear correlations can be measured. The autocorrelation measures linear correlations only.

The delay parameter is found, when using the autocorrelation method, as the first zero passing of the function, if the monotony isn't changing or, otherwise, the first minimum. If the mutual information method is used, the delay parameter is the first minimum of the mutual information function, (Small, 2005).

3. Research methodology

The methodology used in this paper is based on hybrid methodology used in Zhang, (Zhang, 2002). The basic idea is that each time series is composed of a linear and a nonlinear part. The linear part is extracted by fitting an ARIMA model and the nonlinear part is modelled by training an artificial neural network (ANN) with the model residuals time series. The result of modelling both parts can be summed up and the overall prediction performance can be compared with the results obtained taking into consideration the original undecomposed time series modelled only by ANNs.

Figure no 1. An intuitive diagram of nonlinear time series analysis steps.



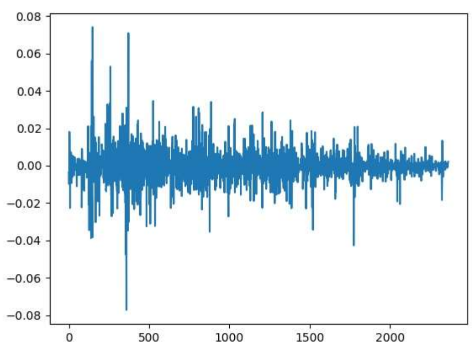
Source: Authors' contribution

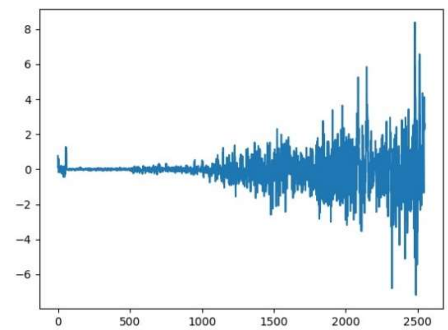
4. Findings

The Cambridge Bitcoin Electricity Consumption Index (CBECI) time series (Figure 2) estimates, using a model developed in 2017, the total amount of electricity needed to mine Bitcoins. The estimation is based on the assumption that the miners are using hardware differing in terms of power that will be used while the profitability threshold is attained. We will not discuss the quality of the model nor the researches suggesting that Bitcoin mining alone could increase the global temperature in a significant manner (Mora, 2018), (Stoll, 2019). The times series corresponds to the daily values of energy consumption during 1.07.204 – 25.06.2021 expressed in GW.

The Euro – Leu time series spans over 15 years, between 2005 and 2015.

*Figure no. **2**. Euro – leu exchange rate time series (left) and CBECI time series (right). The trend was removed and the series were stationarized.*





Source: Authors' contribution

Many models have been constructed for simulating the volatile behavior of the exchange rate. The general idea, prior to the nonlinear time series analysis emergence, is that the exchange rate can be considered as financial assets traded on efficient markets. The exchange rate embeds the available information at the current time and the changes represents the outcome of the unpredictable events. The theory states that an a priori explanation of the exchange rate's evolution is impossible, but an a posteriori explanation could be given.

Nonlinear models and chaos theory are supplying new models for understanding the exchange rate mechanism. The first researches in the fields dates back from 1980. In the majority of papers, it was identified a nonlinear dynamics at the fundament of the exchange rate, including here chaotic dynamics. The literature identifies chaos manifestation in the exchange rates (Federici, 2002), (Brock, 1998), (Pai, (2006), or, on the contrary does not find enough indicators of chaotic dynamics (Brooks, 1998).

The optimum ARIMA model fitted for the exchange rate time series is ARIMA(4,1,1) with the smallest value of the AIC index (-15877.49).The residuals of the model (the nonlinear part) were used to train an ANN. The simplest model of ANN is the multilayer perceptron (MLP) which has the very important property of universal approximation. Our model has one hidden layer with four hidden neurons. The number of neurons in the input layer was set to 8, equal to the value of the embedding dimension. The input layer role is to capture the dynamics of the system. The role of the hidden layer is to discover hidden relationships in the training set.

CBECI time series was modelled using ARIMA(6,1,6) with AIC=5761,69. The neural network trained with the residuals of the ARIMA model was also of MLP type with a similar topology, except that the number of neurons in the hidden layer was set to 5.

The combined predictions from the two models are showing a forecasting capability close to the ANN. The values of the RMSE on the training sets are shown in Table no 1.

*Table no 1**.** Forecasting error obtained with the three methods studied.*

Time series/ Forecasting error (RMSE)	**ARIMA**	**ANN**	**Hybrid**
Leu-euro exchange rate	0.002	0.001387	0.001403
CBECI time series	0.116	0,094	0.082

Source: Authors' contribution

5. Conclusions

Nonlinear behavior may lead to a very intricate and complex time evolution of the nonlinear systems, which may further develop into a chaotic dynamic.

The term "chaotic" is used to describe the aperiodic behavior of an apparently random system. Behind the apparent random dynamics lies in fact the deterministic character of the system whose behavior is completely determined by the equations describing the deterministic chaotic dynamics.

Nonlinearity introduces in fact a better understanding of the complex natural phenomena. Nonlinear dynamics consists of a set of tools and concepts (period doubling, bifurcations, initial conditions sensitivity, attractors, phase space, phase portrait) allowing analyzing the dynamics generated by nonlinear processes.

The hybrid method can be used as an alternate method for predicting future values of time series with performances close to the connectionist algorithms used in machine learning. The ANN used in this study is a basic model which can be fine-tuned by changing the number of layers, the number of neurons on each layer, the activation function. In order to get superior performances for the forecast another ANN model could be employed.

Nonlinear analysis process of the exchange rate time series has shown that the correlation dimension saturates if the embedding dimension is greater that 8. The first minimum of mutual information function is 2, so the delay parameter for the reconstructed state space will be 2. The maximum Lyapunov exponent for m=8 is positive, indicating that the neighboring trajectories diverge exponentially. Plotting the attractor in the reconstructed state space shows dense trajectories with no apparent structure. This could be an indication for high dimensional chaos.

For CBECI time series none of the signatures of chaos could not be detected.

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